

# CCSF PHYC 4D Lecture Notes

Karl M. Westerberg

## Chapter 6

### The Rutherford-Bohr Model of the Atom

Copyright © 2011 by Karl M. Westerberg. Some rights reserved.

This work is licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License. To view a copy of this license, visit

<http://creativecommons.org/licenses/by-nc-sa/3.0/>

or send a letter to

Creative Commons  
444 Castro Street, Suite 900  
Mountain View, CA 94041  
USA

## Thomson model of the atom

- In 1897, Thomson is credited with “discovering” the electron by measuring its charge/mass ratio (essentially using a CRT). He conjectured that these electrons were present in all materials and were much lighter than even the hydrogen atom. These conjectures together with the  $e/m$  measurement earned him the 1906 Nobel Prize.
- Thomson conjectured that an atom consists of  $Z$  electrons mixed together with a ball of positive charge  $+Ze$  (for a net charge of zero), the so-called “plum pudding” model.
- In 1910, Geiger and Marsden (part of Rutherford’s lab) performed an experiment in which alpha particles (helium nuclei) were scattered off of atoms.
- According to the Thomson “plum pudding” model, the maximum scattering from any single atom should be extremely small (about  $0.01^\circ$ ). Even if the alpha particle scatters off of as many as  $10^4$  atoms (target thickness about  $1\text{ }\mu\text{m}$ ), the net scattering angle is expected to average around  $\sqrt{10^4}(0.01^\circ) \approx 1^\circ$ , with an insanely small likelihood of large (greater than  $90^\circ$ ) scattering.
- The results in Rutherford’s lab contradict the large-angle (non-)prediction of the Thomson model.
- Note: Book derivation of scattering off of a Thomson model atom seems suspect to me — I don’t see the justification of ignoring the electrons in the scattering. Since the electrons are very light, and therefore quite mobile (more so than the alpha particle), the atom should polarize significantly when the alpha particle approaches, resulting in a net *attraction* between the atom and the alpha particle. If I were to do the calculation, I would treat the atom as a neutral dielectric ball. In any case, it is hard to deny that such a diffuse distribution of charge is unlikely to deflect the alpha particle by very much. The experimental results obtained by Rutherford’s lab are inconsistent with the Thomson model.

## The Rutherford Model

- Based on the infrequent, but not *overly* infrequent, scattering at large angles, Rutherford proposed that the positive charge does not lie in some diffuse ball, but instead lies in a very small, dense nucleus. The nucleus is *much* smaller than the atom. Large scattering angles occur when the scattered particle happens to get very close to one of the nuclei.
- Imagine that a small positive charge (charge  $+ze$ , mass  $m$ ) encounters a fixed nucleus (charge  $+Ze$ , mass  $\gg m$ ), and scatters off of it.

$v_0$  = velocity of  $+ze$  charge far away from nucleus

$b$  = impact parameter

$h$  =  $v_0 b$  = angular momentum/ $m$

$\epsilon$  =  $\frac{1}{2}v_0^2$  = total energy/ $m$

$\vec{a} = \frac{A}{r^2} \hat{r} \quad A = \frac{zZ}{m} \frac{e^2}{4\pi\epsilon_0}$

- Solving  $d^2\vec{r}/dt^2 = A\hat{r}/r^2$  using polar coordinates, and solving for  $r$  as a function of  $\theta$ , one gets

$$\frac{1}{r} = C \cos \theta - \frac{A}{h^2} \quad C = \sqrt{\frac{A^2}{h^4} + \frac{2\epsilon}{h^2}}$$

This is the equation for a conic section with eccentricity  $e = Ch^2/|A|$ . This result applies to any attractive ( $A < 0$ ) or repulsive ( $A > 0$ ) inverse-square law force, as well as bound ( $\epsilon < 0$ ) or unbound ( $\epsilon \geq 0$ ) orbits. The angle  $\theta = 0$  is the point of closest approach.

- For unbound orbits, the path is hyperbolic with  $r \rightarrow \infty$  when  $\theta = \pm\theta_0$ , where

$$\tan \theta_0 = \frac{h\sqrt{2\epsilon}}{A} = \frac{mv_0^2 b}{zZ} \left( \frac{e^2}{4\pi\epsilon_0} \right)^{-1}$$

The scattering angle  $\phi = \pi - 2\theta_0$ , and so  $\tan \theta_0 = \cot(\frac{1}{2}\phi)$ .

- Solving for  $b$  as a function of  $\phi$  gives

$$b = \frac{zZ}{mv_0^2} \frac{e^2}{4\pi\epsilon_0} \cot(\frac{1}{2}\phi) = b_0 \cot(\frac{1}{2}\phi)$$

where  $b_0 = (zZ/mv_0^2)(e^2/4\pi\epsilon_0)$  is the impact parameter corresponding to  $90^\circ$  scattering.

- Further details for the above calculation can be found in a separate text, titled “Inverse Square Forces”.
- Let  $f(\phi)$  be the fraction of charges that scatter through an angle greater than  $\phi$ . That occurs when the impact parameter is less than the corresponding value of  $b$ . For a sample of  $N$  total atoms distributed over an area  $A$  and thickness  $l$  (number density  $n = N/Al$ ):

$$f(\phi) = (\pi b^2/A)N = \pi b^2 n l = \pi b_0^2 n l \cot^2(\tfrac{1}{2}\phi)$$

- Let  $N(\phi) d\Omega$  be the fraction of particles scattered into a given solid angle  $d\Omega$  at scattering angle  $\phi$ . ( $d\Omega = dA/R^2$ , where  $dA$  is the area of the detector and  $R$  is the distance from the scattering event to the detector). Then

$$\begin{aligned} N(\phi) &= \frac{df}{d\Omega} = \frac{f(\phi) - f(\phi + d\phi)}{(R d\phi)(2\pi R \sin \phi)/R^2} \\ &= \frac{1}{2\pi \sin \phi} \left( -\frac{df}{d\phi} \right) \\ &= \frac{\pi b_0^2 n l}{4\pi \sin(\tfrac{1}{2}\phi) \cos(\tfrac{1}{2}\phi)} 2 \cot(\tfrac{1}{2}\phi) \tfrac{1}{2} \csc^2(\tfrac{1}{2}\phi) \\ &= \frac{n l b_0^2}{4 \sin^4(\tfrac{1}{2}\phi)} \end{aligned}$$

- Geiger and Marsden verified the above formula in a series of experiments detailed in the book.

## Line Spectra

- There is a very brief history written up in Wikipedia under the title “Spectrum analysis”.
- Solids, liquids, and high pressure gasses tend to emit continuous spectra (e.g., blackbody). The key reason for the continuous spectrum is the complicated interaction among the  $\approx 10^{23}$  atoms/molecules.
- In contrast, very hot low pressure (weakly or non-interacting) gasses tend to emit discrete lines.

- The existence of discrete spectral lines was known by the early to mid-1800's. By the mid-1800's, several scientists had measured spectral emission lines for several different elements, including hydrogen. Each element seemed to have its own signature emission spectrum.
- In 1885, Balmer discovered that some of the hydrogen emission spectral lines were well described by a formula

$$\lambda = \lambda_{\text{limit}} \frac{n^2}{n^2 - n_0^2}$$

where  $n > n_0 = 2$ . The formula was later generalized to include other integer values of  $n_0$ .  $\lambda_{\text{limit}}$  depends on  $n_0$  and is called the *series limit*, the (lower) limiting wavelength that results when  $n \rightarrow \infty$ .

Series	$n_0$	$\lambda_{\text{limit}}$ (nm)	spectrum
Lyman	1	91.1	UV
Balmer	2	364.5	Visible
Paschen	3	820.1	IR
Brackett	4	1458.0	Far IR
Pfund	5	2278.2	Far IR

- A few years later, Rydberg reworked the result:

$$\frac{1}{\lambda} = \frac{n_0^2}{\lambda_{\text{limit}}} \left( \frac{1}{n_0^2} - \frac{1}{n^2} \right)$$

The combination  $n_0^2/\lambda_{\text{limit}}$  is actually a universal constant, now called the Rydberg constant with the present-day accept value of

$$R_{\infty} = \frac{me^4}{64\pi^3\epsilon_0^2\hbar^3c} = 1.097 \times 10^7 \text{ m}^{-1}$$

- Another curious fact, known as the “Ritz combination principle” (1908??), is that some of the frequencies from the emission spectrum add up to produce other frequencies in the emission spectrum. This principle is built into Rydberg’s formula.
- Ultimately, the physical explanation of emission spectra comes down to the emission of photons of energy  $h\nu = E_i - E_f$  when an electron makes a transition from one energy level ( $E_i$ ) to another ( $E_f$ ).

## Bohr Model (1913)

- Bohr developed this model in 1913 as a followup to Rutherford's model of the atom. He postulated that electrons occupy circular orbits around the positive nucleus, like planets around a sun.
- Consider the hydrogen atom (one electron orbiting one proton). For orbits of radius  $r$ :

$$F = \frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2} = \frac{m_e v^2}{r}$$

$$K = \frac{1}{2} m_e v^2 = \frac{1}{2} F r = \frac{1}{4\pi\epsilon_0} \frac{e^2}{2r}$$

$$U = -\frac{1}{4\pi\epsilon_0} \frac{e^2}{r} = -2K$$

$$E = K + U = -K = -\frac{1}{4\pi\epsilon_0} \frac{e^2}{2r} < 0$$

- Normally, accelerating charges radiate electromagnetic radiation. This would reduce  $E$  towards  $-\infty$ , and cause the electron to spiral in towards the nucleus. Bohr cleverly avoided this trap by asserting that it doesn't happen for certain special orbits, namely those for which the angular momentum is an integer multiple of  $\hbar$  (I'm guessing that he was looking ahead and trying to explain the hydrogen emission spectrum).

$$L = |\vec{r} \times \vec{p}| = m_e v r = n\hbar$$

- Only certain orbits are allowed.

$$K = \frac{1}{2} m_e v^2 = \frac{L^2}{2m_e r^2} = \frac{1}{4\pi\epsilon_0} \frac{e^2}{2r}$$

Solving for  $r$  yields

$$r_n = \frac{4\pi\epsilon_0 L^2}{m_e e^2} = \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2} n^2 = a_0 n^2$$

Plugging in values yields  $a_0 = 0.0529 \text{ nm}$ , the Bohr radius of the hydrogen atom.

- Since the radius is quantized, the total energy is too:

$$E_n = -\frac{1}{4\pi\epsilon_0} \frac{e^2}{2r} = -\frac{m_e e^4}{2(4\pi\epsilon_0)^2 \hbar^2} \frac{1}{n^2} = -\frac{E_0}{n^2}$$

Plugging in the numbers yields  $E_0 = 13.6 \text{ eV}$ , the binding energy of the hydrogen atom.

- Line spectra can now be explained. When an electron drops from a state of energy  $E_{n_1}$  down to a state of energy  $E_{n_2}$  ( $n_1 > n_2$ ), it releases a photon whose energy is  $E_{n_1} - E_{n_2}$ .

$$h\nu = E_{n_1} - E_{n_2} = E_0 \left( \frac{1}{n_1^2} - \frac{1}{n_2^2} \right)$$

Since  $\nu = c/\lambda$ , this can be rewritten

$$\frac{1}{\lambda} = \frac{E_0}{hc} \left( \frac{1}{n_1^2} - \frac{1}{n_2^2} \right) = R_\infty \left( \frac{1}{n_1^2} - \frac{1}{n_2^2} \right)$$

Plugging in numbers yields  $R_\infty = 1.097 \times 10^7 \text{ m}^{-1}$ , which is consistent with the observed hydrogen atom spectrum.

- A photon can also be absorbed by an electron in a state of energy  $E_{n_2}$ , causing it to make a transition to a state of energy  $E_{n_1}$ . The photon frequency must be given by

$$h\nu = E_{n_1} - E_{n_2}$$

This defines the *absorption spectrum*.

- On the surface, it appears that the absorption spectrum and emission spectrum should be identical. However, atoms rarely occupy energy levels  $n_2$  that are different from 1, so only transitions  $1 \rightarrow n_1$  show up in the absorption spectrum.
- More direct evidence of energy levels: the Franck-Hertz experiment. Electrons are accelerated through a potential  $V$  while travelling through a low pressure gas. When an electron encounters an atom of the gas, there is a possibility that it might collide and lose  $\Delta E$  (energy gap between the ground state and an excited state of the atom). When  $eV$  is a multiple of  $\Delta E$ , it is possible for a given electron to lose virtually

all of its kinetic energy, and not be able to pass through a modest grid potential  $V_0$  to get to the collector. Experimental results show a drop in current as  $V$  passes through integer multiples of  $\Delta E/e$ , revealing the discrete nature of the energy levels of the atoms in the gas (results for mercury gas are shown in Fig. 6.22[24] on p. 189[196] of the textbook).

- Correspondance Principle: Quantum mechanics must agree with classical mechanics in the limit where classical physics is known to be valid by experiment (i.e., for “large” systems). This makes sense: previous experimental evidence of classical mechanics is still valid.
  - In relation to the Bohr model, classical behavior should be evident as  $n \rightarrow \infty$ .  $r_{10000} = 5.29 \text{ mm}$ .
  - As  $n \rightarrow \infty$ , energy levels, allowed radii, and allowed frequencies in the emission spectrum all become continuous
  - The emission frequency for  $n \rightarrow n - 1$  transition matches the classically predicted frequency, which is the same as the frequency of the electron orbit itself (see textbook).
- What Bohr got right.
  - Angular momentum of electrons (and other particles) is quantized.
  - Atomic energy levels are quantized.
  - Allowed frequencies of EM radiation are related to the allowed atomic energy levels ( $h\nu = E_i - E_f$ ).
  - The allowed energy values for hydrogen (and any atom/ion with just one electron present), and corresponding EM spectrum.
  - The (average) radii of the electrons in those states.
- What Bohr got wrong.
  - Electrons do not execute circular orbits around the nucleus. In fact, the electrons exist as a probability standing wave around the nucleus — there is no “path” followed by an electron.
  - Allowed angular momentum values are incorrect ( $|\vec{L}| = \sqrt{l(l+1)}\hbar$ , where  $l = 0, 1, 2, \dots$ ).



- Association between angular momentum and principle quantum number (which defines energy) is also incorrect — in general there are multiple angular momentum values allowed for each possible energy value.
- No prediction of transition rates and probabilities.
- Unable to handle more than one electron effectively (Bohr model can be a decent approximation if there is one *valence* electron, such as with sodium).
- According to de Broglie, an electron exists as a wave whose wavelength is given by  $\lambda = h/p$ . If the electron wave is to exist in a stable orbit of radius  $r$ , then it should exist as a standing wave. This requires

$$2\pi r = n\lambda = nh/m_e v$$

It follows that

$$L = m_e v r = nh/2\pi = n\hbar$$